Central limit theorem: *If the sample size is large enough, then the sampling distribution for the mean is (approximately) normal. Specifically*

$$\overline{x} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right),$$

where μ is the population mean and σ is the population standard deviation.

Implication: If the sample size n is big enough and $\overline{x}$ is the mean of a simple random sample, then

(*) $P(-E < \overline{x} - \mu < E) = P\left(-\frac{E}{\sigma/\sqrt{n}} < z < \frac{E}{\sigma/\sqrt{n}}\right)$;

(*) If $E_\alpha = z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$, then $\frac{E_\alpha}{\sigma/\sqrt{n}} = z_{\alpha/2}$ and

$$P(-E_\alpha < \overline{x} - \mu < E_\alpha) = P(-z_{\alpha/2} < z < z_{\alpha/2}) = 1 - \alpha.$$

Conclusion: *If the population standard deviation σ is known*, then

$$\overline{x} \pm E_\alpha = (\overline{x} - E_\alpha, \overline{x} + E_\alpha)$$

is a $(1 - \alpha) \times 100\%$ confidence interval for μ. E.g.,

$$\alpha = 0.05 \implies 95\%\text{-confidence} \quad \text{and} \quad \alpha = 0.01 \implies 99\%\text{-confidence}$$

Comments:

(i) If the original population has a normal distribution, then

$$\overline{x} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

for ***any*** sample size n.

(ii) If the original population has a distribution which is *'normal-like'* (one mode, symmetric, no outliers), then $n > 30$ is typically large enough (rule-of-thumb).

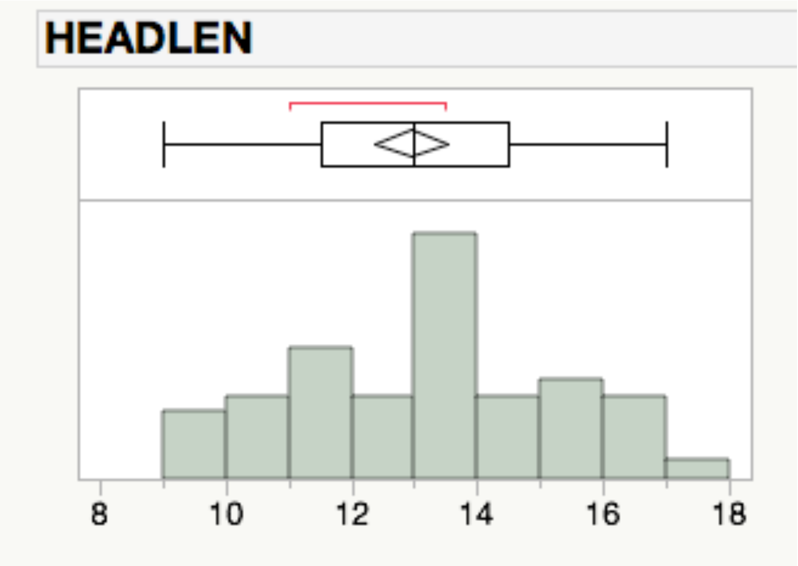
(iii) If the original population has a distribution that is *far* from normal, then the sample size might need to be considerably larger for the methods we use here to be reliable.

Example. A (simple random) sample of $n = 54$ bears. The lengths of their heads were recorded (in inches). Construct a 95% confidence interval for the mean length of bears heads in the population.

(*) $\overline{x} = 12.954$

(*) Population standard deviation: $\sigma = 2.152$.

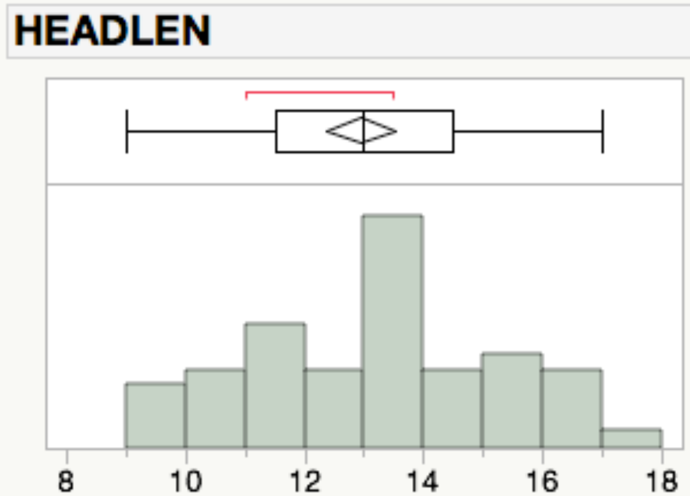
(*) Data appears to come from a normal-like distribution:



(*) Margin of error: $E_{0.05} = z_{0.025} \cdot \frac{\sigma}{\sqrt{n}} = 1.96 \cdot \frac{2.152}{\sqrt{54}} \approx 0.574$

(*) Confidence interval: $\overline{x} \pm E_\alpha = 12.954 \pm 0.574 = (12.38, 13.528)$.

More complete JMP output:



Summary Statistics	
Mean	12.953704
Std Dev	2.1440319
Std Err Mean	0.2917658
Upper 95% Mean	13.538912
Lower 95% Mean	12.368496
N	54

(*) The reported standard deviation is the *sample* SD $s \approx 2.144$.

(*) $\frac{s}{\sqrt{n}}$ (or $\frac{\sigma}{\sqrt{n}}$) is called the *standard error for the mean* (SE).

(*) The reported SE is smaller than the one we calculated (because $\sigma < s$)...

(*) ... But the reported confidence interval $(12.368, 13.539)$ is ***wider*** than the one we calculated..?

Central limit theorem, again: *If the sample size is large enough, then the sampling distribution for the mean is (approximately) normal. Specifically*

$$\overline{x} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right) \implies \frac{\overline{x} - \mu}{\sigma} \sim N(0, 1),$$

where μ *is the population mean and* σ *is the population standard deviation.*

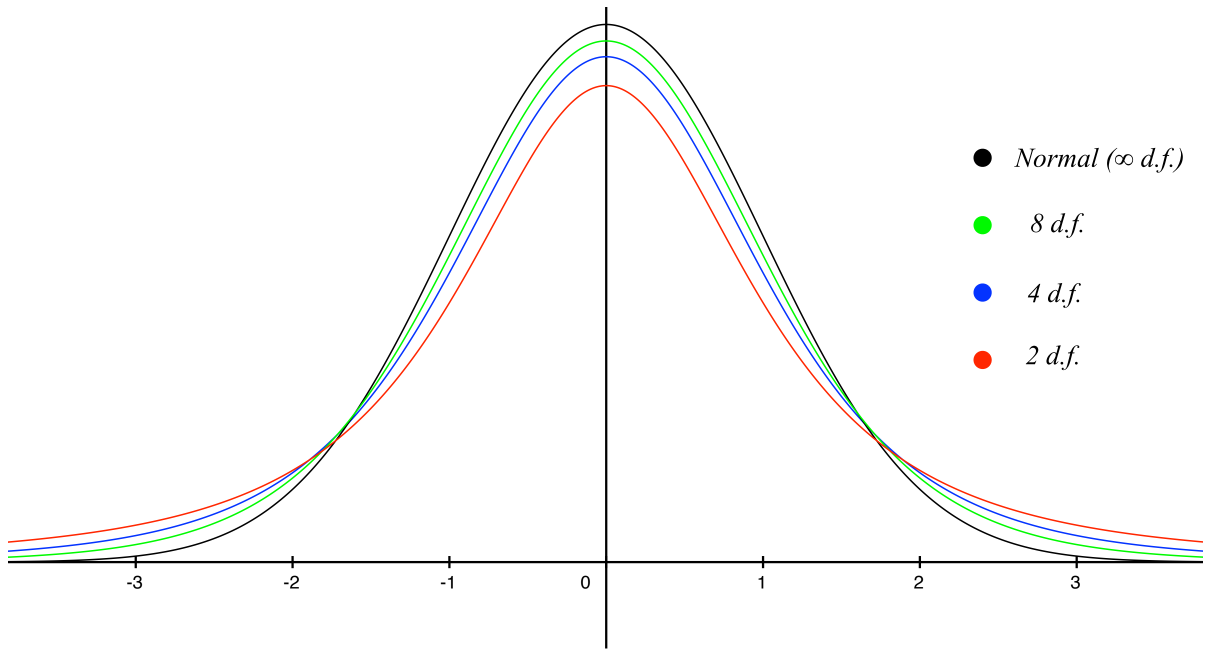
(*) If the population standard deviation (σ) is ***not*** known (the usual case!), we use sample standard deviation (s) instead.

(*) **Complication**: $\dfrac{\overline{x} - \mu}{s/\sqrt{n}}$ does ***not*** follow $N(0, 1)$.

(*) **Resolution:** $\dfrac{\overline{x} - \mu}{s/\sqrt{n}}$ ***does*** follow *Student's* t-distribution, with $n - 1$ *degrees of freedom.*

(*) There is a separate t-distribution for each number of degrees of freedom (d.f.)

(*) All t-distributions are 'bell-shaped', but shorter than and with thicker tails than $N(0,1)$.



(*) As $d.f. \to \infty$, the t-distribution approaches $N(0,1)$.

Conclusion: If $n > 30$ or the population has a normal distribution, then $\frac{\overline{x} - \mu}{s/\sqrt{n}}$ follows the t-distribution, with $n - 1$ *degrees of freedom.*

(*) The t-table is arranged to make it easy to find ***critical*** values for specific areas under the curve for (many) different d.f.

TABLE A-3 *t* Distribution: Critical *t* Values

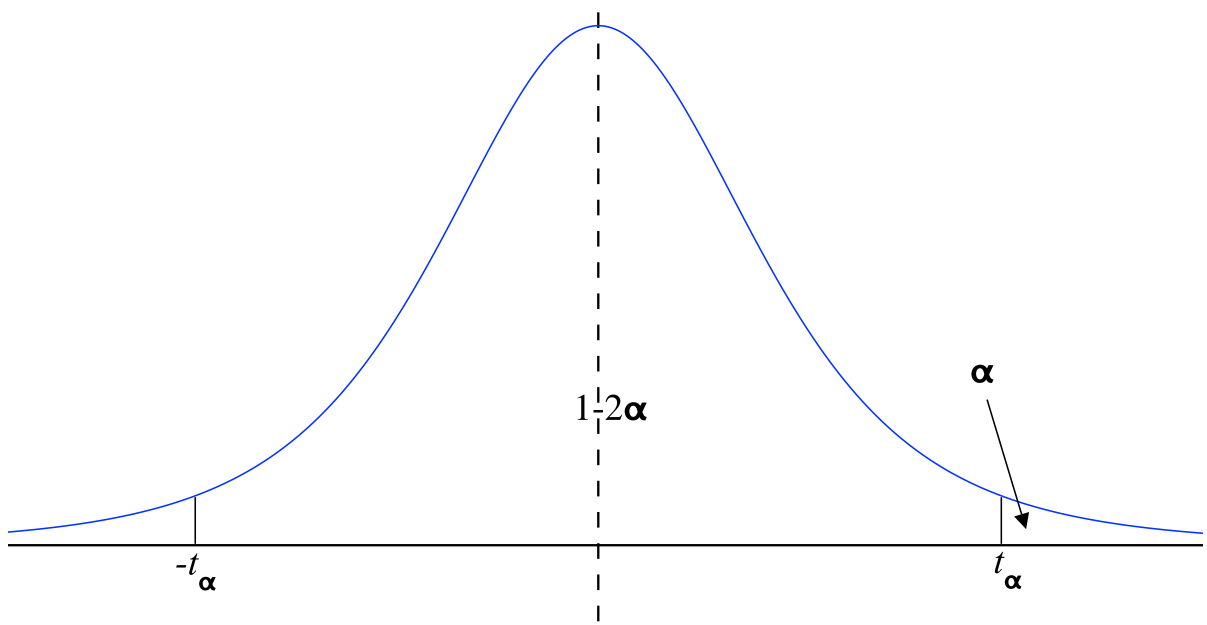
	Area in One Tail				
	0.005	0.01	0.025	0.05	0.10
	Area in Two Tails				
Degrees of Freedom	0.01	0.02	0.05	0.10	0.20
1	63.657	31.821	12.706	6.314	3.078
2	9.925	6.965	4.303	2.920	1.886
3	5.841	4.541	3.182	2.353	1.638
4	4.604	3.747	2.776	2.132	1.533
5	4.032	3.365	2.571	2.015	1.476
6	3.707	3.143	2.447	1.943	1.440
7	3.499	2.998	2.365	1.895	1.415
8	3.355	2.896	2.306	1.860	1.397
9	3.250	2.821	2.262	1.833	1.383
10	3.169	2.764	2.228	1.812	1.372
11	3.106	2.718	2.201	1.796	1.363
12	3.055	2.681	2.179	1.782	1.356
13	3.012	2.650	2.160	1.771	1.350
14	2.977	2.624	2.145	1.761	1.345
15	2.947	2.602	2.131	1.753	1.341
16	2.921	2.583	2.120	1.746	1.337
17	2.898	2.567	2.110	1.740	1.333
18	2.878	2.552	2.101	1.734	1.330
19	2.861	2.539	2.093	1.729	1.328
20	2.845	2.528	2.086	1.725	1.325
21	2.831	2.518	2.080	1.721	1.323
22	2.819	2.508	2.074	1.717	1.321
23	2.807	2.500	2.069	1.714	1.319
24	2.797	2.492	2.064	1.711	1.318
25	2.787	2.485	2.060	1.708	1.316
26	2.779	2.479	2.056	1.706	1.315
27	2.771	2.473	2.052	1.703	1.314
28	2.763	2.467	2.048	1.701	1.313
29	2.756	2.462	2.045	1.699	1.311
30	2.750	2.457	2.042	1.697	1.310

(*) Same principle as before: the critical value t_α (m d.f.) is the number such that the area in one tail is

$$P(t > t_\alpha) = \alpha$$

This also means that the area between the two tails is

$$P(-t_\alpha < t < t_\alpha) = 1 - 2\alpha.$$



For example, if $m = 25$ and $\alpha = 0.025$, then $t_\alpha = 2.060$

(*) If $E_\alpha = t_{\alpha/2} \cdot \frac{s}{\sqrt{n}}$, then

$$P(|\overline{x} - \mu| < E_\alpha) = P\left(\left|\frac{\overline{x} - \mu}{s/\sqrt{n}}\right| < t_{\alpha/2}\right) = 1 - \alpha$$

I.e., if $\alpha = 0.025$, then

$$\overline{x} \pm E_{0.05} = \overline{x} \pm t_{0.025} \cdot \frac{s}{\sqrt{n}}$$

is a 95% confidence interval for the population mean μ.

Example. 95% Confidence interval for bear head length.

(*) $\overline{x} = 12.954$, $s = 2.144$, $n = 54$.

(*) $n = 54 \implies d.f. = 54 - 1 = 53.$

	Area in One Tail 0.005	0.01	0.025	0.05	0.10
Degrees of Freedom	Area in Two Tails 0.01	0.02	0.05	0.10	0.20
50	2.678	2.403	2.009	1.676	1.299
55	2.668	2.396	2.004	1.673	1.297

(*) $t_{0.025} \approx 2.006$ (3/5 of the way from 50 d.f. to 55 d.f.)

(*) $E_{0.05} = t_{0.025} \cdot \frac{s}{\sqrt{n}} \approx 2.006 \cdot \frac{2.144}{\sqrt{54}} \approx 0.585$

(*) 95% confidence interval:

$$\overline{x} \pm E_{0.05} = 12.954 \pm 0.585 = (12.369, 13.539).$$