# Hypothesis test for a population proportion.

**Requirements:**

**(i)** Simple random sample.

**(ii)** Conditions for *binomial distribution* (and normal approximation) are satisfied:

- A fixed number ($n =$ sample size) of independent trials.
- Two outcomes for each trial, $S$(uccess) and $F$(ailure), with constant probability $P(S) = p$.
- The $\#(S) > 5$ and $\#(F) > 5$.

**Procedure:**

**1.** Formulate Null and Alternative hypotheses:

$$H_0 : p = p_0$$
$$H_a : p < p_0 \quad \text{(left tail)} \quad \text{OR}$$
$$H_a : p > p_0 \quad \text{(right tail)} \quad \text{OR}$$
$$H_a : p \neq p_0 \quad \text{(two tails)}$$

**2.** Choose significance level $\alpha = P(\text{type I error})$

(Almost always $\alpha = 0.05$ or $\alpha = 0.01$.)

**3.** Calculate (observed) test statistic:

$$z^* = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 \cdot q_0}{n}}},$$

where $\hat{p}$ = sample proportion, $q_0 = 1 - p_0$ and $n$ = sample size.

**Distribution of test statistic:** $N(0, 1)$.

**4.** Decision Criterion.

(a) *Traditional method:* Reject $H_0$ if ...

$$H_a: \; p < p_0 \implies z^* < -z_\alpha$$
$$H_a: \; p > p_0 \implies z^* > z_\alpha$$
$$H_a: \; p \neq p_0 \implies |z^*| > z_{\alpha/2}$$

(b) *$P$-value method:* Reject $H_0$ if ...

$$H_a: \; p < p_0 \implies P = P(z < z^*) < \alpha$$
$$H_a: \; p > p_0 \implies P = P(z > z^*) < \alpha$$
$$H_a: \; p \neq p_0 \implies P = P(|z| > |z^*|) < \alpha$$

**5.** Summarize conclusions.

## Hypothesis test for a population mean $\mu$ (population standard deviation *not* known).

**Requirements:**

**(i)** Simple random sample.

**(ii)** Population is (approximately) normally distributed,
or sample size $n > 30$,
or both.

**Procedure:**

**1.** Formulate Null and Alternative hypotheses:

$$H_0 : \mu = \mu_0$$
$$H_a : \mu < \mu_0 \quad \text{(left tail)} \quad \text{OR}$$
$$H_a : \mu > \mu_0 \quad \text{(right tail)} \quad \text{OR}$$
$$H_a : \mu \neq \mu_0 \quad \text{(two tails)}$$

**2.** Choose significance level $\alpha = P(\text{type I error})$

(Almost always $\alpha = 0.05$ or $\alpha = 0.01$.)

**3.** Calculate (observed) test statistic:

$$t^* = \frac{\overline{x} - \mu_0}{s/\sqrt{n}},$$

where $\overline{x}$ = sample mean, $s$ = sample s.d. and $n$ = sample size.

**Distribution of test statistic:** Student's $t$-distribution with $n-1$ degrees of freedom.

**4.** Decision Criterion.

(a) *Traditional method:* Reject $H_0$ if ...

$$H_a: \mu < \mu_0 \implies t^* < -t_\alpha$$
$$H_a: \mu > \mu_0 \implies t^* > t_\alpha$$
$$H_a: \mu \neq \mu_0 \implies |t^*| > t_{\alpha/2}$$

(b) *$P$-value method:* Reject $H_0$ if ...

$$H_a: \mu < \mu_0 \implies P = P(t < t^*) < \alpha$$
$$H_a: \mu > \mu_0 \implies P = P(t > t^*) < \alpha$$
$$H_a: \mu \neq \mu_0 \implies P = P(|t| > |t^*|) < \alpha$$

**Comment:** Statistical software computes $P$-values for $t$-distributions. If you are relying on a $t$-table for your hypothesis tests, use the traditional (critical value) method.

**5.** Summarize conclusions.

**Example.** A lobbyist for the cereal industry asserts that the mean *sugar content* of all cereals is less than 0.3 grams of sugar per gram of cereal. A simple random sample of $n = 35$ cereal boxed is collected and their sugar contents are observed. Summary statistics: $\overline{x} = 0.282$ and $s = 0.197$.

*Does the data support the lobbyist's claim?*

(*) Hypotheses: $H_0 : \mu = 0.3$ and $H_a : \mu < 0.3$.

(*) Significance level: $\alpha = 0.05$.

(*) Critical value ($\alpha = 0.05$, 34 d.f., one tail): $t_c = 1.691$.

(*) Critical region: $t < -1.691$ (left tail test).

(*) Test statistic: $t^* = \dfrac{\overline{x} - \mu_0}{s/\sqrt{n}} = \dfrac{0.282 - 0.3}{0.197/\sqrt{35}} \approx -0.54$

(*) Fail to reject $H_0$ because $-1.691 < -0.54$

**Conclusion:** There is not sufficient evidence to support the claim that the mean sugar content of all cereals is less than 0.3 grams of sugar per gram of cereal.

## Hypothesis tests for two population proportions.

*Usually, we will be testing the null hypothesis* $H_0 : p_1 = p_2$.

**Requirements:**

**(i)** Two *independent* simple random samples.

**(ii)** Conditions for *binomial distribution* (and normal approximation) are satisfied for both.

(*) Two samples are ***independent*** if the observations in one sample are not related to the observations in the other in any systematic way. (This does not mean that we remove an observation from one sample because it happens to be related, by pure chance, to an observation from the other one.)

(*) If the samples are independent, then the sample proportions are *independent random variables*. The ***variance*** of a *sum or difference* of independent random variables is equal to the sum of their variances.

(*) The sum or difference of two normal random variables is normal.

**Notation.**

- $p_1$ and $p_2$ — proportions in populations 1 and 2, respectively.
- $n_1$ and $n_2$ — sizes of the samples taken from populations 1 and 2, respectively.
- $x_1$ and $x_2$ — number of *S(uccesses)* in samples taken from populations 1 and 2, respectively.
- $\hat{p}_1 = \dfrac{x_1}{n_1}$ and $\hat{p}_2 = \dfrac{x_2}{n_2}$ (sample proportions).
- $\hat{q}_1 = 1 - \hat{p}_1$ and $\hat{q}_2 = 1 - \hat{p}_2$.
- $\overline{p} = \dfrac{x_1 + x_2}{n_1 + n_2}$: the *pooled estimate* of $p_1$ and $p_2$.
- $\overline{q} = 1 - \overline{p}$.

**Procedure:**

**1.** Formulate Null and Alternative hypotheses:

$$H_0 : p_1 = p_2$$
$$H_a : p_1 < p_2 \quad \text{(left tail)} \quad \text{OR}$$
$$H_a : p_1 > p_2 \quad \text{(right tail)} \quad \text{OR}$$
$$H_a : p_1 \neq p_2 \quad \text{(two tails)}$$

**2.** Choose significance level $\alpha = P(\text{type I error})$

(Almost always $\alpha = 0.05$ or $\alpha = 0.01$.)

**3.** Calculate (observed) test statistic:

$$z^* = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\dfrac{\overline{p} \cdot \overline{q}}{n_1} + \dfrac{\overline{p} \cdot \overline{q}}{n_2}}} = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\dfrac{\overline{p} \cdot \overline{q}}{n_1} + \dfrac{\overline{p} \cdot \overline{q}}{n_2}}}$$

**Distribution of test statistic:** $N(0, 1)$.

**4.** Decision Criterion.

(a) *Traditional method:* Reject $H_0$ if ...

$$H_a:\ p_1 < p_2 \implies z^* < -z_\alpha$$
$$H_a:\ p_1 > p_2 \implies z^* > z_\alpha$$
$$H_a:\ p \neq p_0 \implies |z^*| > z_{\alpha/2}$$

(b) *P-value method:* Reject $H_0$ if ...

$$H_a:\ p < p_0 \implies P = P(z < z^*) < \alpha$$
$$H_a:\ p > p_0 \implies P = P(z > z^*) < \alpha$$
$$H_a:\ p_1 \neq p_2 \implies P = P(|z| > |z^*|) < \alpha$$

**5.** Summarize conclusions.

**Example.** In clinical test of the adverse effects of Viagra, of 734 men in a treatment group, 7% experienced *dyspepsia*. In contrast, 2% of the 725 men in the control group experienced dyspepsia.

*Does dyspepsia occur at a higher rate among Viagra users than among nonusers?*

(*) $p_1$=proportion of Viagra users that experience dyspepsia.

(*) $p_2$ = proportion of nonusers that experience dyspepsia.

$$H_0 : p_1 = p_2 \qquad H_a : p_1 > p_2.$$

(*) Significance level: $\alpha = 0.05$.

(*) $0.07 \cdot 734 = 51.38 \implies x_1 = 51, \quad 0.02 \cdot 725 = 14.5 \implies x_2 = 15.$

$$\text{Pooled estimate: } \overline{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{66}{1459} \approx 0.045.$$

(*) Test statistic:

$$z^* = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\frac{\overline{p} \cdot \overline{q}}{n_1} + \frac{\overline{p} \cdot \overline{q}}{n_2}}} = \frac{0.05}{\sqrt{\frac{0.045 \cdot 0.955}{734} + \frac{0.045 \cdot 0.955}{725}}} \approx 4.6$$

(*) $P$-value: This is a right-tailed test (because $H_a : p_1 > p_2$, i.e., $H_a : (p_1 - p_2) > 0$) so $P = P(z > z^*)$:

$$P = P(z > 4.6) < 0.0001$$

(*) Reject $H_0$ because $P < 0.05$.

*Conclusion:* There is sufficient evidence to support the claim that Dyspepsia occurs at a higher rate among Viagra users.

*Confidence interval for the difference of two population proportions.*

(*) Same notation as before.

(*) Point estimate for $p_1 - p_2$: $\hat{p}_1 - \hat{p}_2$.

(*) Margin of error: $E_\alpha = z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}_1 \hat{q}_1}{n_1} + \frac{\hat{p}_2 \hat{q}_2}{n_2}}$

(*) Confidence interval for $p_1 - p_2$:

$$(\hat{p}_1 - \hat{p}_2) \pm E_\alpha$$